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HEAT TRANSFER INTO A SEMI-INFINITE REGION WITH A BOUNDARY MOVING ACCORDING TO AN ARBITRARY LAW

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We present an improvement to the previously given [1] method of solving certain problems with a moving boundary. We shall investigate an unsteady heat transfer in a semi-infinite region with a moving boundary at a specified temperature and zero initial conditions.

Using a coordinate system attached to the moving boundary, we describe the process in the form of the following problem:

$$\left[\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}-u(t)\frac{\partial}{\partial x}\right]T=0,\quad 0\leqslant x\leqslant\infty,\quad 0<t<\infty$$

$$T\,|_{x=0}=T_0(t),\quad T\,|_{x=\infty}=0,\quad T\,|_{t=0}=0$$

where u is the velocity of the moving boundary. The only quantity to be determined is the temperature gradient at the boundary of the region $q_0=(\partial T\,/\,dx)_{x=0}$.

In contrast to [1] we make the substitution

$$T=\theta\exp\int_0^t\frac{1}{4}u^2\,dt \tag{1}$$

We now obtain the following problem for θ :

$$\left[\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}-u(t)\frac{\partial}{\partial x}-\frac{u^2(t)}{4}\right]\theta=0,\qquad 0\leqslant x<\infty,\quad 0<t<\infty$$

$$\theta\,|_{x-0}=\theta_0=T_0\exp\int_0^t\frac{1}{4}u^2dt \tag{2}$$

Equation (2) can be written in the form [2]

$$\left(M-\frac{\partial}{\partial x}\right)\left(L+\frac{\partial}{\partial x}\right)\theta=0 \tag{3}$$

$$M = \sum_{m=0}^{\infty} b_m(t) \frac{\partial^{(1-m)/2}}{\partial t^{(1-m)/2}}, \quad L = \sum_{n=0}^{\infty} a_n(t) \frac{\partial^{(1-n)/2}}{\partial t^{(1-n)/2}} \tag{4}$$

$$\frac{d^\nu f(t)}{dt^\nu} = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t f(\tau)(t-\tau)^{-\nu} d\tau, \quad -\infty < \nu < 1 \tag{5}$$

It was shown before [1, 2] that instead of Eq. (2), we can consider an equation formed by the right-hand side factor of the operator in (3)

$$\left(L + \frac{\partial}{\partial x}\right)\theta = 0 \tag{6}$$

all solutions of which satisfy automatically the condition at infinity. Rewriting (6) for $x = 0$ we obtain the expression which yields a solution to the problem in question

$$-(\partial\theta / \partial x)_{x=0} = L\theta_0(t) \tag{7}$$

The function θ satisfies (6) and (2) simultaneously, therefore differentiating (6) with respect to x and eliminating $\partial\theta / \partial x$ and $\partial^2\theta / \partial x^2$ from (2) we obtain an equation determining the operator L

$$[L^2 - u(t)L + {}^1/_4 u^2(t)]\theta = \partial\theta / \partial t \tag{8}$$

Let us substitute into this equation L from (4) and transform the expressions in such a manner, that they will act only on the function to be determined. Equating the factors accompanying the similar derivatives, we obtain

$$a_0 = 1, \quad a_1 = \frac{u}{2}, \quad a_2 = 0, \quad a_3 = -\frac{u'}{8}, \quad a_4 = 0, \quad a_5 = \frac{u''}{16} \tag{9}$$

$$a_6 = -\frac{5}{128} u'^2, \quad a_7 = -\frac{5}{128} u''', \quad a_8 = \frac{25}{512} u'u'', \quad a_9 = \frac{7}{256} u^{\mathrm{IV}} -$$

$$\frac{15}{512} u'^3, \quad a_{10} = -\frac{109}{1024} u'u''' - \frac{13}{256} u''^2, \quad a_{11} = \frac{175}{1024} u'^2 u'' - \frac{21}{1024} u^{\mathrm{V}}$$

$$a_{12} = -\frac{1105}{2^{15}} u'^4 + \frac{1757}{2^{13}} u''u''' + \frac{887}{2^{13}} u'u^{\mathrm{IV}}$$

. . . .

$$\sum_{m=0}^{m+2r \leqslant s+1} \sum_{r=0} \binom{\frac{1-m}{2}}{r} a_m a_{s+1-m-2r}^{(r)} - u a_s = 0, \quad s \geqslant 0$$

. . . .

The expression sought for $q_0(t)$ is now given by the formulas (7), (1), (4) and (9), and it is much simpler than that obtained in the version given in [1]. When $u = \mathrm{const}$, the series in (4) and (7) contains just two terms

$$-q_0(t) = \exp\left(-\frac{1}{4}u^2 t\right)\left(\frac{d^{1/2}}{dt^{1/2}} + \frac{u}{2}\right)\exp\left(\frac{1}{4}u^2 t\right) T_0(t)$$

Let us consider in detail the case $u = \alpha t + \beta$ (α and β are constants) which was investigated before by the method of separation of variables [3]. Equation (8) is satisfied by the operator

$$L = \frac{1}{2}(\alpha t + \beta) + \sum_{n=0}^{\infty} c_n \alpha^n \frac{\partial^{1/2 - 3/2 n}}{\partial t^{1/2 - 3/2 n}}, \quad c_n - \mathrm{const} \tag{10}$$

where

$$c_0 = 1, \quad c_1 = -{}^1/_8, \quad c_2 = -5/128, \quad c_3 = -15/512, \quad c_4 = -1105/2^{15} \tag{11}$$

$$c_5 = -1695/2^{15}, \quad c_6 = -414125/2^{22}, \quad c_7 = -59025/2^{18}$$

. . . .

$$2c_s = -\left[1 - \frac{3}{4}s\right] c_{s-1} - \sum_{n=1}^{s-1} c_n c_{s-n}, \quad s \geqslant 1$$

We note that all c_n, $n \geqslant 1$ are negative.

Let us establish the convergence of the series (4) for the present example. From (11) we find by induction that $c_n c_{s-n} > c_{n-1} c_{s-n+1}$ if $n > s - n$. Therefore replacing each term in the sum by the maximum value of $c_{s-1} c_1 = -c_{s-1} / 8$, we obtain

$$|c_s| < {}^7/_{16} s \, |c_{s-1}| \tag{12}$$

Using the mean value theorem we can obtain from (5) the following estimate:

$$\left| \frac{\partial^\nu \theta}{\partial t^t} \right| \leqslant \Gamma^{-1}(1-\nu)\, t^\nu \sup|\theta|, \quad \nu < 0 \tag{13}$$

Substitution of (12) and (13) into (4) yields a majorizing series the general term of which has the form

$$({}^7/_{16})^n \Gamma(n) \Gamma^{-1}[{}^3/_2(n+1)]\, t^{(3n+1)/2}$$

and from this the convergence of (4) follows.

Using (12), (13) and the properties of the gamma-function, we can obtain an estimate for the error, provided that in the series (10) all terms up to $n = N$ are preserved. For a nondecreasing $\theta_0(t)$ the quantity $(-\partial\theta / \partial x)_{x=0} > 0$ exceeds the exact value by the quantity $\delta > 0$, and

$$\delta < |c_N| \sup \theta_0 \cdot (7\pi^{1/2}/8) 2^N 3^{-(3N/2)-2} \Gamma^{-1}(N)$$
$$\Gamma[{}^3/_2(N+1)]\Gamma^{-1}[{}^3/_2(N+1)+{}^1/_2]\Gamma[{}^1/_2(N+1)+{}^1/_2]\Gamma^{-1}[{}^1/_2(N+1)+{}^2/_3]$$
$$\Gamma[{}^1/_2(N+1)+{}^1/_6]\Gamma^{-1}[{}^1/_2(N+1)+{}^1/_3]\Gamma[{}^1/_2(N+1)+1]\Gamma^{-1}[{}^1/_2(N+1)+{}^7/_6]$$
$$\left[\Gamma^{-1}\left(\frac{N+2}{2}\right) + \Gamma\left(\frac{N+2}{2}\right)\Gamma^{-1}\left(\frac{N+3}{2}\right)\cdot\frac{7}{8}\left(\frac{t}{3}\right)^{3/2}\right] t^{(3N+4)/2} \exp\frac{49t^3}{576}$$

At small t the formula (7) is more suitable for computations than the formulas given in [3].

If the function u is specified in the form of a series in exponential functions

$$u = \sum_{n=0}^{\infty} \beta_n e^{-n\alpha t}, \quad \alpha, \beta = \text{const}$$

then the operator L can be found from (8) in the form of an expansion suitable for computations at large t

$$L = \sum_{n=0}^{\infty} e^{-n\alpha t} P_n, \quad P_0 = \frac{\partial^{1/2}}{\partial t^{1/2}}, \quad P_n\theta(t) = \int_0^t K_n(t-\tau)\,\theta(\tau)\,d\tau, \quad n > 0$$

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